Synthesis of a High Frequency Reactance

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FOSTER'S reactance theorem[1] provides the necessary information for the synthesis of the most general dissipationless reactance provided the frequency range is such that lumped reactances, namely inductance coils and condensers, are available. At the higher frequencies these are difficult to obtain practically and the impedance elements must be treated in design as the transmission lines with distributed inductance and capacity which they truly are. Accordingly, it is convenient to have general methods for the synthesis of a high frequency reactance presented in terms of the number, arrangement, and constants of the transmission line reactance elements. Two such methods applying to dissipationless reactances over a definite, limited frequency band, as most commonly needed in practice, are given in the following.

The most general pure reactance, no matter what the frequency may be, must according to the Foster theorem have the form shown by curve of reactance *versus* frequency of Fig. 1. Since only the form of the curve is intended to be pictured here, the abscissa might equally well have been δ instead of frequency, where δ is proportional to the deviation in frequency from some base frequency, say f_0, and is most conveniently defined by:

$$\delta=((f-f_0)/f_0)\pi/2.$$

Suppose that in a certain frequency band, say between $\delta=\delta_0$ to $\delta=\delta_{2n}$ where $1\gg\delta_{2n}\gg\delta_0$, a reactance is desired which shall have zeros at $\delta=\delta_1,\ \delta=\delta_3,\ \cdots,\ \delta=\delta_{2n-1}$ and infinities at $\delta=\delta_0,\ \delta=\delta_2,\ \cdots,\ \delta=\delta_{2n}$.

One form of function which will meet these specifications and also possess everywhere a positive slope as does the curve of Fig. 1 is

$$Z=jA\delta\frac{(\delta^2-\delta_1{}^2)(\delta^2-\delta_3{}^2)\cdots(\delta^2-\delta^2{}_{2n-1})}{(\delta^2-\delta_0{}^2)(\delta^2-\delta_2{}^2)\cdots(\delta^2-\delta^2{}_{2n})},\tag{1}$$

in which A is a negative, real constant. Expanding by partial fractions

$$Z=jA\delta\left[\frac{B_0}{\delta^2-\delta_0{}^2}+\frac{B_2}{\delta^2-\delta_2{}^2}+\cdots+\frac{B_{2n}}{\delta^2-\delta^2{}_{2n}}\right],\tag{2}$$

in which

$$B_m=\frac{(\delta_m{}^2-\delta_1{}^2)(\delta_m{}^2-\delta_3{}^2)\cdots(\delta_m{}^2-\delta^2{}_{2n-1})}{(\delta_m{}^2-\delta_0{}^2)(\delta_m{}^2-\delta_2{}^2)\cdots(\delta_m{}^2-\delta^2{}_{m-2})(\delta_m{}^2-\delta^2{}_{m+2})\cdots(\delta_m{}^2-\delta^2{}_{2n})}$$

and is seen to be positive. Thus a group of impedances of the form $jAB_m\delta/(\delta^2-\delta_m{}^2)$, connected in series will give the desired reactance. One way to realize an impedance of this form is with two transmission lines, properly chosen and connected in parallel.

The impedance looking into a transmission line, shorted at its far end, of characteristic impedance Z_1, and one-quarter of a wave-length long at the frequency f_0 is[2]

$$Z_{\text{in}}'=jZ_1\tan\theta,\tag{3}$$

which is very nearly equal to $-j\dfrac{Z_1}{\delta}$.

$$\left(\theta=\frac{\pi}{2}\left[1+\frac{f-f_0}{f_0}\right]=\frac{\pi}{2}+\delta\right.$$

so that $\sin\theta$ is very close to unity and $\cos\theta$ is practically equal to $-\delta$ for small values of δ).

[1] R. M. Foster, "A Reactance Theorem," Bell Sys. Tech. J. **5**, 259 (1924); or E. A. Guillemin, *Communication Networks* (John Wiley and Sons, 1935), Vol. II.

[2] See, for instance, W. E. Everitt, *Communication Engineering* (McGraw-Hill, 1937), Chap. V.

The impedance looking into a transmission line, open circuited at its far end,[3] of characteristic impedance Z_2, and one-quarter of a wavelength long at the frequency f_0 is

$$Z_{\text{in}}''=jZ_2\delta. \tag{4}$$

The resultant impedance of the two lines in parallel is:

$$Z_m=-j\frac{\delta Z_1}{\delta^2-Z_1/Z_2}. \tag{5}$$

So if $Z_{1,\,m}$ is made equal to $-AB_m$ and $Z_{2,\,m}$ made equal to $Z_{1,\,m}/\delta_m{}^2$, then n such parallel combinations in series will suffice to provide the desired reactance.

Since the choice of the zero and infinity points, δ_1, δ_2, etc., has been purely arbitrary, it is possible to employ the inverse network of that just obtained as a satisfying form for the needed reactance. The inverse network's reactance will have zeros wherever the previous network had infinities, and vice versa. The new network will consist of a group of n admittances connected in parallel[4] each of the form of Z_m in Eq. (5). It can easily be checked that an admittance of this form results from Z_{in}' and Z_{in}'' connected in series. Thus a group of n such series combinations connected in parallel will also give the desired reactance.

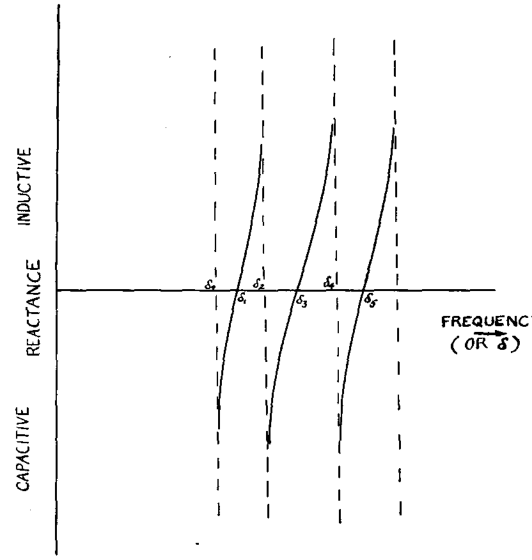

FIG. 1.

Both networks give reactances which are symmetrical about the point $\delta=0$. In some cases this may be advantageous; in other cases it is simply necessary to place the operating frequency band entirely above or entirely below $f=f_0$.

[3] If it is undesirable to use an open-ended line, then a shorted, half-wave line may be used instead, for it has an input impedance of this same form.

[4] This by virtue of the well-known inverse impedance theorem which states that the inverse impedance of a group of impedance elements connected in series is obtained by connecting the inverses of the impedance elements in parallel.